Exercise 38

If H is the Heaviside function defined in Example 2.2.6, prove, using Definition 2, that $\lim_{t\to 0} H(t)$ does not exist. [*Hint:* Use an indirect proof as follows. Suppose that the limit is L. Take $\varepsilon = \frac{1}{2}$ in the definition of a limit and try to arrive at a contradiction.]

Solution

In Example 2.2.6, the Heaviside function is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}.$$

Suppose that the limit of $H(t)$ as $t \to 0$ exists and that it's equal to L.

$$\lim_{t\to 0} H(t) = L$$

According to Definition 2, this is logically equivalent to

$$\text{if} \quad 0 < |t - 0| < \delta \qquad \text{then} \qquad |H(t) - L| < \varepsilon$$

for all positive ε. Choose $\varepsilon = \frac{1}{2}$.

$$\text{if} \quad |t| < \delta \qquad \text{then} \qquad |H(t) - L| < \frac{1}{2}$$

Consider this statement at two different times for which the hypothesis is the same, $t = -0.1$ and $t = 0.1$, for example.

$$\begin{cases} \text{At } t = -0.1, \quad \text{if} \quad 0.1 < \delta \qquad \text{then} \qquad |0 - L| < \frac{1}{2} \\ \text{At } t = 0.1, \quad \text{if} \quad 0.1 < \delta \qquad \text{then} \qquad |1 - L| < \frac{1}{2} \end{cases}$$

Simplify each consequent.

$$\begin{aligned} |0 - L| &< \frac{1}{2} \\ |L| &< \frac{1}{2} \\ -\frac{1}{2} < L &< \frac{1}{2} \end{aligned} \qquad\qquad \begin{aligned} |1 - L| &< \frac{1}{2} \\ -\frac{1}{2} < 1 - L &< \frac{1}{2} \\ -\frac{3}{2} < -L &< -\frac{1}{2} \\ \frac{3}{2} > L &> \frac{1}{2}. \end{aligned}$$

These two consequents are contradictory because one says that L is between -0.5 and 0.5, and the other says that L is between 0.5 and 1.5. Therefore, $\lim_{t\to 0} H(t)$ does not exist.